The Law of Large Numbers

If X_1 , X_2 , . . . , X_N are pairwise independent, identically distributed random variables with expectation E(X) and variance V(X) then:

the probability that $\left|\frac{X_1 + X_2 + \ldots + X_N}{N} - E(X)\right| \leq \varepsilon$

is greater than $1 - \frac{V(X)}{N\varepsilon^2}$ where ε is a positive number.

Proof:

We will apply Chebyshev's Inequality to the random variable

$$\frac{X_1 + X_2 + \ldots + X_N}{N} \quad .$$

First we find the expectation and variance of this random variable using the formulas from my paper on the algebra of expectations.

Since the expectation of a sum of random variables equals the sum of their expectations we get: $E(X_1+X_2+ \ldots +X_N) = NE(X)$.

Since the expectation of a constant times a random variable is the product of the constant times the expectation of the random variable

we get: $E\left(\frac{X_1 + X_2 + \ldots + X_N}{N}\right) = 1/N \times NE(X) = E(X)$.

Since the variance of the sum of pairwise independent random variables is the sum of their variances we get:

$$V(X_1+X_2+ \ldots +X_N) = NV(X) .$$

Since the variance of a constant times a random variable is equal to the constant squared times the variance of the random variable we get:

$$V\left(\frac{X_1 + X_2 + \ldots + X_N}{N}\right) = \frac{NV(X)}{N^2} = \frac{V(X)}{N}$$

Chebyshev's Inequality says that:

the probability that $|Y - E(Y)| \leq \varepsilon$ is greater than $1 - \frac{V(Y)}{\varepsilon^2}$.

So since $E\left(\frac{X_1 + X_2 + \ldots + X_N}{N}\right) = E(X)$ and $V\left(\frac{X_1 + X_2 + \ldots + X_N}{N}\right) = \frac{V(X)}{N}$,

we get when substituting $\left(\frac{X_1 + X_2 + \ldots + X_N}{N}\right)$ for Y in Chebyshev's Inequality:

the probability that $\left|\frac{X_1 + X_2 + \ldots + X_N}{N} - E(X)\right| \leq \varepsilon$

is greater than $1 - \frac{V(X)}{N\varepsilon^2}$

□

Proof of Bernoulli's Theorem using Chebyshev's Law of Large Numbers

Let $X_i = 1$ if there is a success on the ith trial
Let $X_i = 0$ if there is a failure on the ith trial

let p be the probability of a success on the ith trial
let $q = 1 - p$ be the probability of a failure on the ith trial

then $E(X_i) = 1 \times p + 0 \times q = p$

and $V(X_i) = (1 - p)^2 p + (0 - p)^2 q = q^2 p + p^2 q = qp(q+p) = pq$

So by Chebyshev's law of large numbers we have:

the probability that $\left|\frac{X_1 + X_2 + \ldots + X_N}{N} - p\right| \leq \varepsilon$ is greater than $1 - \frac{pq}{N\varepsilon^2}$.

Let M be the actual number of successes in N trials.
$M = X_1 + X_2 + \ldots + X_N$ so we get:

the probability that $\left|\frac{M}{N} - p\right| \leq \varepsilon$ is greater than $1 - \frac{pq}{N\varepsilon^2}$.

Let η be a positive number, then
if N is sufficiently large, the probability that $\left|\frac{M}{N} - p\right| \leq \varepsilon$
is greater than $1 - \eta$.

This is Bernoulli's Theorem.

□

Comment

In Bernoulli's example, p = 3/5 , q = 2/5 , e = 1/50, and

$\frac{pq}{N\varepsilon^2}$ = 1/1001 . Solving for N we get N = 600,600 .

Using Bernoulli's formula, NT is 25,550 where Bernoulli's NT is our N. So Bernoulli's formula gives a much better value for N.

If we modify Bernoulli's example so that C is 9 instead of 1000, then NT becomes 13,900 and using Chebyshev's inequality the number of trials becomes 6,000. So when C is small, Chebyshev's inequality can give a smaller number of trials than Bernoulli's formulas.

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