The Law of Large Numbers

If X_1 , X_2 , ..., X_N are pairwise independent, identically distributed random variables with expectation $E(X)$ and variance $V(X)$ then:

\n The Law of Large Numbers\n If *X*₁, *X*₂, …, *X*_N are pairwise independent, identically distributed\n random variables with expectation E(X) and variance V(X) then:\n the probability that
$$
\left| \frac{X_1 + X_2 + ... + X_N}{N} - E(X) \right| \leq \varepsilon
$$
\n is greater than\n $1 - \frac{V(X)}{N\varepsilon^2}$ \n where ε is a positive number.\n Proof:\n We will apply Chebyshev's Inequality to the random variable\n $\frac{X_1 + X_2 + ... + X_N}{N}$ \n .\n First we find the expectation and variance of this random variable\n

Proof:

We will apply Chebyshev's Inequality to the random variable

$$
\frac{X_1 + X_2 + \dots + X_N}{N} \quad .
$$

First we find the expectation and variance of this random variable using the formulas from my paper on the algebra of expectations.

Since the expectation of a sum of random variables equals the sum of their expectations we get: $E(X_1+X_2+...+X_N) = NE(X)$.

Since the expectation of a constant times a random variable is the product of the constant times the expectation of the random variable $\frac{1}{N}$
 $\frac{1}{N}$
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we get:
$$
E\left(\frac{X_1 + X_2 + ... + X_N}{N}\right) = 1/N \times NE(X) = E(X)
$$
.

Since the variance of the sum of pairwise independent random variables is the sum of their variances we get:

$$
V(X_1+X_2+...+X_N)=NV(X).
$$

Since the variance of a constant times a random variable is equal to the constant squared times the variance of the random variable we get: Since the variance of the sum of pairwise independent random
variables is the sum of their variances we get:
 $V(X_1+X_2+...+X_N) = NV(X)$.
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 $\frac{(X)}{N}$ variables is the sum of their variances we get:
 $V(X_1+X_2+...+X_N) = NV(X)$.

Since the variance of a constant times a random variable is equal to

the constant squared times the variance of the random variable we

get:
 $V\left(\frac$ Since the variance of a constant times a random variable is equal to
the constant squared times the variance of the random variable we
get:
 $V\left(\frac{X_1 + X_2 + ... + X_N}{N}\right) = \frac{NV(X)}{N^2} = \frac{V(X)}{N}$
Chebyshev's Inequality says that: variance of a constant times a random variable is equal to
 $\left(\frac{X_2 + ... + X_N}{N}\right) = \frac{NV(X)}{N^2} = \frac{V(X)}{N}$
 $\left(\frac{X_1 + X_2 + ... + X_N}{N}\right) = \frac{NV(X)}{N}$
 $\left(\frac{X_1 + X_2 + ... + X_N}{N}\right) = E(X)$ and $V\left(\frac{X_1 + X_2 + ... + X_N}{N}\right) = \frac{V(X)}{N}$,
 $\left(\frac{X_1 +$ of a constant times a random variable is equal to

d times the variance of the random variable we
 $\int \frac{N V(X)}{N^2} = \frac{V(X)}{N}$

ality says that:
 $|Y - E(Y)| \le \varepsilon$ is greater than $1 - \frac{V(Y)}{\varepsilon^2}$.
 $\frac{N+X_N}{N} = E(X)$ and $V\left$ random variable is equal to

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ter than $1 - \frac{V(Y)}{\varepsilon^2}$.
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V\left(\frac{X_1 + X_2 + ... + X_N}{N}\right) = \frac{NV(X)}{N^2} = \frac{V(X)}{N}
$$

Chebyshev's Inequality says that:

the probability that $|Y - E(Y)| \le \varepsilon$ is greater than $1 - \frac{V(Y)}{c^2}$.

So since
$$
E\left(\frac{X_1 + X_2 + ... + X_N}{N}\right) = E(X)
$$
 and $V\left(\frac{X_1 + X_2 + ... + X_N}{N}\right) = \frac{V(X)}{N}$,

get:
 $V\left(\frac{X_1 + X_2 + ... + X_N}{N}\right) = \frac{NV(X)}{N^2} = \frac{V(X)}{N}$

Chebyshev's Inequality says that:

the probability that $|Y - E(Y)| \le \varepsilon$ is greater than $1 - \frac{V(Y)}{\varepsilon^2}$.

So since $E\left(\frac{X_1 + X_2 + ... + X_N}{N}\right) = E(X)$ and $V\left(\frac{X_1 + X_2 + ...$ $\left(\frac{N}{N}\right)$ for Y in Chebyshev's $\frac{X}{2} = \frac{V(X)}{N}$

ys that:

Y)| $\leq \varepsilon$ is greater than $1 - \frac{V(Y)}{\varepsilon^2}$.
 $= E(X)$ and $V\left(\frac{X_1 + X_2 + ... + X_N}{N}\right) = \frac{V(X)}{N}$,
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 $E(Y)| \le \varepsilon$ is greater than $1 - \frac{V(Y)}{\varepsilon^2}$.
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 $\frac{1}{N} + \frac{1}{N} + \frac{X_2 + ... + X_N}{N} - E(X) \le \varepsilon$ eater than $1 - \frac{V(Y)}{\varepsilon^2}$.
 $V\left(\frac{X_1 + X_2 + ... + X_N}{N}\right) = \frac{V(X)}{N}$,
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 $\left| -E(X) \right| \le \varepsilon$ at:
 ϵ is greater than $1 - \frac{V(Y)}{\epsilon^2}$.
 (X) and $V\left(\frac{X_1 + X_2 + ... + X_N}{N}\right) = \frac{V(X)}{N}$,
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 $\frac{X_2 + ... + X_N}{N} - E(X) \le \epsilon$ $|E(Y)| \le \varepsilon$ is greater than $1 - \frac{V(Y)}{\varepsilon^2}$.
 $\left(\frac{K_N}{N}\right) = E(X)$ and $V\left(\frac{X_1 + X_2 + ... + X_N}{N}\right) = \frac{V(X)}{N}$,
 $\arg \left(\frac{X_1 + X_2 + ... + X_N}{N}\right)$ for Y in Chebyshev's
 $\left|\frac{X_1 + X_2 + ... + X_N}{N} - E(X)\right| \le \varepsilon$
 $\frac{(X)}{N\varepsilon^2}$

Chebyshev's Inequality says that:
\nthe probability that
$$
|Y - E(Y)| \le \varepsilon
$$
 is greater than $1 - \frac{V(Y)}{\varepsilon^2}$.
\nSo since $E\left(\frac{X_1 + X_2 + ... + X_N}{N}\right) = E(X)$ and $V\left(\frac{X_1 + X_2 + ... + X_N}{N}\right) = \frac{V(X)}{N}$,
\nwe get when substituting $\left(\frac{X_1 + X_2 + ... + X_N}{N}\right)$ for Y in Chebyshev's
\nInequality:
\nthe probability that $\left|\frac{X_1 + X_2 + ... + X_N}{N} - E(X)\right| \le \varepsilon$
\nis greater than $1 - \frac{V(X)}{N\varepsilon^2}$

Proof of Bernoulli's Theorem using Chebyshev's Law of Large Numbers

Let $X_i = 1$ if there is a success on the ith trial Let $X_i = 0$ if there is a failure on the ith trial

let p be the probability of a success on the ith trial let $q = 1 - p$ be the probability of a failure on the ith trial

then $E(X_i) = 1$ x p + 0 x q = p

and $V(X_i) = (1 - p)^2 p + (0 - p)^2 q = q^2 p + p^2 q = qp(q+p) = pq$

So by Chebyshev's law of large numbers we have:

Let $X_i = 1$ if there is a success on the ith trial

Let $X_i = 0$ if there is a failure on the ith trial

let p be the probability of a success on the ith trial

let q = 1 - p be the probability of a failure on the ith tria $N \qquad \qquad \begin{array}{c} | \ \ \text{if} \ \text{if}$ $\frac{n+X_N}{N} - p \leq \varepsilon$ is greater than $1 - \frac{pq}{Nc^2}$. success on the ith trial
failure on the ith trial
obability of a failure on the ith trial
 $0 \times q = p$
 $+(0-p)^2 q = q^2 p + p^2 q = qp(q+p) = pq$
w of large numbers we have:
 $\frac{q_1 + X_2 + ... + X_N}{N} - p \le \varepsilon$ is greater than $1 - \frac{pq}{N\varepsilon^2}$.
ther $N\varepsilon^2$ Let M be the actual number of successes in N trials. $M = X_1 + X_2 + ... + X_N$ so we get: bability of a failure on the ith trial
 $0 \times q = p$
 $(0-p)^2 q = q^2 p + p^2 q = qp(q+p) = pq$

of large numbers we have:
 $\frac{+X_2 + ... + X_N}{N} - p \le \varepsilon$ is greater than $1 - \frac{pq}{N\varepsilon^2}$.

ber of successes in N trials.
 $- p \le \varepsilon$ is greater tha qp(q+p) = pq

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ter than $1 - \frac{pq}{N\varepsilon^2}$.

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 $1 - \frac{pq}{N\varepsilon^2}$.
 $- p \le \varepsilon$

the probability that $\left|\frac{M}{N} - p\right| \leq \varepsilon$ is greater than $1 - \frac{pq}{N^2}$. $N \begin{bmatrix} 1 \end{bmatrix}^{-\alpha}$ is given the $N\varepsilon^2$ pq $N\varepsilon^2$

Let η be a positive number, then if N is sufficiently large, the probability that $\left|\frac{M}{N}-p\right| \leq \varepsilon$ $N \begin{bmatrix} 1 \end{bmatrix}$ is greater than $1 - \eta$

This is Bernoulli's Theorem.

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 \Box

Comment

In Bernoulli's example, $p = 3/5$, $q = 2/5$, $e = 1/50$, and

 $\frac{pq}{N}$ = 1/1001. Solving for N we get N = 600,600. $N\epsilon^2$ 1.1001. Borting to

Using Bernoulli's formula, NT is 25,550 where Bernoulli's NT is our N. So Bernoulli's formula gives a much better value for N.

If we modify Bernoulli's example so that C is 9 instead of 1000, then NT becomes 13,900 and using Chebyshev's inequality the number of trials becomes 6,000. So when C is small, Chebyshev's inequality can give a smaller number of trials than Bernoulli's formulas.

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